[14] The author is indebted for technical assistance in the preparation of these matings to W. Gencarella.

[15] No definite case of a double crossover in the $y - spl$ region was observed.

[16] Two verified crossovers between y and sc were detected as y^2 *sc w spl* phenotypes.

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ON THE NUMBER OF BOUND STATES IN A CENTRAL FIELD OF FORCE

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1. The present note contains some fairly elementary remarks concerning the number of bound state solutions of the Schrödinger equation

$$\nabla^2\psi + E\psi = V(r)\psi$$

for a central field of force, more specifically, the number n_l of bound state solutions of the radial wave equation

$$\phi'' - l(l+1)r^{-2}\phi + E\phi = V(r)\phi \tag{1}$$

for angular momentum l. We assume the integral

$$I = \int_0^\infty r\,|V(r)|\,dr \tag{2}$$

to be *finite*, and we wish to estimate n_l in terms of I. (In the units chosen V has the dimension (length)$^{-2}$, so that I is dimensionless.) R. Jost and A. Pais (ref. 1, p. 844) have shown that no bound states occur if $I < 1$. Our aim is to derive the more general inequality

$$(2l+1)n_l < I \tag{3}$$

(equality excluded). The number n_l counts the distinct stationary energy values corresponding to equation (1). If the $(2l + 1)$-fold degeneracy of each of them is taken into account it is seen that for a given angular momentum l there are less than I bound states, and no bound states occur if $l \geq {}^1/_2(I - 1)$. The estimate (3) is best possible in the sense that for a given l potentials may be constructed which have a prescribed number n_l of bound states for that angular momentum and for which I approaches

$(2l + 1)n_l$ arbitrarily closely (see section 5 below). The whole question is taken up because the finiteness of I plays a significant role in several recent investigations on scattering theory.[1-4] (V may have any singularities consistent with a finite value of I.)

2. As is well known, n_l is the number of zeros (not counting $r = 0$) of that solution $\phi(r)$ of the equation

$$\phi'' - l(l + 1)r^{-2}\phi = V(r)\phi \tag{4}$$

($E = 0$) which vanishes at the origin. Special care must be taken with a possible bound state $E = 0$. Since I is finite any solution of (4) has the following asymptotic behavior at infinity. The expression $r^{-(l+1)}\phi(r)$ always approaches a finite limit, say λ, as $r \to \infty$. If $\lambda \neq 0$, $\phi(r)$ increases indefinitely. If $\lambda = 0$ the expression $r^l\phi(r)$ approaches a finite limit μ, and $\mu \neq 0$. In the latter case $\phi(r)$ is square integrable if $l > 0$, and accordingly $E = 0$ is a bound state.[5] For $l = 0$, $E = 0$ is never a bound state if I is finite. For the purpose of our discussion, however, we shall count $r = \infty$ as a zero of $\phi(r)$—even if $l = 0$—whenever $\lim_{r\to\infty} r^{-(l+1)}\phi(r) = 0$, and interpret the inequality (3) accordingly.[6]

Replace in equation (1) $V(r)$ by a potential $V_1(r)$ such that $V_1(r) \leq V(r)$ for all r, and denote by n_l' the number of bound states for the new potential. Then $n_l' \geq n_l$. We shall choose $V_1(r) = -W(r)$, where $W(r) = |V(r)|$, and study the equation

$$\phi'' - l(l + 1)r^{-2}\phi = -W(r)\phi \qquad W(r) = |V(r)|. \tag{5}$$

Denote by $\nu_1, \nu_2, \ldots, \nu_n$ $(n = n_l')$ the successive zeros of $\phi(r)$ $(0 < \nu_1 < \nu_2 < \ldots < \nu_n)$, and set $\nu_0 = 0$. We shall prove

$$\int_\alpha^\beta rW(r)\,dr > 2l + 1; \; \alpha = \nu_{k-1}, \beta = \nu_k, k \geq 1. \tag{6}$$

The inequality (3) is obtained by adding the n inequalities (6), for we find then

$$I = \int_0^\infty rW(r)\,dr \geq \int_0^{\nu_n} rW(r)\,dr > n_l'(2l + 1) \geq n_l(2l + 1).$$

3. Preliminary Remarks on $\phi(r)$.—The solution of equation (5) which vanishes at the origin is uniquely determined up to a constant factor. As $r \to 0$, $r^{-(l+1)}\phi(r)$ approaches a finite non-vanishing limit κ. Choosing $\kappa = 1$, we find from (5)

$$\phi(r) = r^{l+1} - \int_0^r G(r, \rho)\phi(\rho)W(\rho)\,d\rho. \tag{7}$$

$\phi(r)$ is then real. Here $G(r, \rho)$ is the fundamental solution of the equation $f'' - l(l + 1)r^{-2}f = 0$, i.e., $\partial^2G(r, \rho)/\partial r^2 - l(l + 1)r^{-2}G(r, \rho) = 0$, $G(r, r) = 0$, and $\partial G(r, \rho)/\partial r = 1$ for $r = \rho$. We have

$$G(r, \rho) = (2l + 1)^{-1}H(r, \rho); \quad H(r, \rho) = r(r/\rho)^l - \rho(\rho/r)^l \quad (8)$$

if $r > 0$, $\rho > 0$. Clearly $H(r, \rho) > 0$ if $r > \rho$. Since, for $r \to 0$, $r^{-(l+1)}\phi(r) \to 1$, the integral in (7) is absolutely convergent.

In the sequel we shall need the inequality

$$H(\beta, \rho)H(\rho, \alpha) \leq \rho(H(\beta, \alpha) - Y(\beta, \alpha))$$
$$Y(\beta, \alpha) = 2(\alpha\beta)^{1/2}[1 - (\alpha/\beta)^{l+1/2}] > 0 \qquad (\beta > \alpha). \qquad (9)$$

To derive it consider $Z(\rho, \beta, \alpha) = \rho H(\beta, \alpha) - H(\beta, \rho)H(\rho, \alpha)$. By straight forward computation

$$\begin{aligned} Z(\rho, \beta, \alpha) &= \rho(\alpha\beta)^{1/2}\{(\rho^2/\alpha\beta)^{l+1/2} + (\alpha\beta/\rho^2)^{l+1/2} - 2(\alpha/\beta)^{l+1/2}\} \\ &= \rho(\alpha\beta)^{1/2}\{[(\rho/\sqrt{\alpha\beta})^{l+1/2} - (\sqrt{\alpha\beta}/\rho)^{l+1/2}]^2 + 2[1 - (\alpha/\beta)^{l+1/2}]\} \\ &\geq \rho Y(\beta, \alpha) \end{aligned}$$

which establishes (9).

4. *Proof of* (6).—We distinguish four cases[7] according as $\alpha = 0$, $\alpha > 0$; $\beta < \infty$, $\beta = \infty$.

(*a*) $\alpha = 0$, $\beta = \nu_1 < \infty$. On the open interval $(0, \beta)$ ϕ is positive, and hence, by (7), $\phi(r) \leq r^{l+1}$. Since $\phi(\beta) = 0$ we have from (7) and (8)

$$\begin{aligned} (2l + 1)\beta^{l+1} &= \int_0^\beta H(\beta, \rho)\phi(\rho)W(\rho)d\rho \leq \int_0^\beta H(\beta, \rho)\rho^{l+1}W(\rho)\, d\rho \\ (2l + 1)\beta^{l+1} &\leq \beta^{l+1}[\int_0^\beta \rho W(\rho)\, d\rho - \int_0^\beta (\rho/\beta)^{2l+1}\rho W(\rho)\, d\rho]. \end{aligned}$$

On dividing by β^{l+1} we find (6) because the last integral is positive.

(*b*) $\alpha > 0$, $\beta < \infty$. Since $\phi(\alpha) = 0$, the derivative $\phi'(\alpha)$ does not vanish. If we replace $\phi(r)$ by $\chi(r) = \phi(r)/\phi'(\alpha)$, then $\chi(\alpha) = 0$, $\chi'(\alpha) = 1$, and hence

$$\chi(r) = G(r, \alpha) - \int_\alpha^r G(r, \rho)\chi(\rho)W(\rho)\, d\rho \qquad (10)$$

On the interval $\alpha < r < \beta$, therefore, $0 < \chi(r) \leq G(r, \alpha)$. Thus, for $r = \beta$,

$$G(\beta, \alpha) = \int_\alpha^\beta G(\beta, \rho)\chi(\rho)W(\rho)\, d\rho \leq \int_\alpha^\beta G(\beta, \rho)G(\rho, \alpha)W(\rho)\, d\rho,$$

or

$$\begin{aligned} (2l + 1)H(\beta, \alpha) &\leq \int_\alpha^\beta H(\beta, \rho)H(\rho, \alpha)W(\rho)\, d\rho \\ &\leq [H(\beta, \alpha) - Y(\beta, \alpha)] \int_\alpha^\beta \rho W(\rho)\, d\rho \end{aligned}$$

[see (9)]. Division by $H(\beta, \alpha)$ establishes (6).

(*c*) $\alpha = 0$, $\beta = \nu_1 = \infty$. Here we use (7), and, as in case (*a*), $r^{l+1} \geq \phi(r) > 0$ for all positive r. By assumption, $\tau(r) = (2l + 1)r^{-(l+1)}\phi(r)$ approaches 0 as $r \to \infty$. By (7),

$$2l + 1 - \tau(r) - \int^r K(r, \rho)\phi(\rho)W(\rho)\, d\rho = 0$$

where

$$K(r, \rho) = r^{-(l+1)}H(r, \rho) = \rho^{-l}(1 - (\rho/r)^{2l+1}) < \rho^{-l}$$

Hence

$$\int_0^r \rho W(\rho)\, d\rho \geq \int_0^r \rho^{l+1}K(r, \rho)W(\rho)\, d\rho \geq 2l + 1 - \tau(r) + \int_0^r K(r, \rho)[\rho^{l+1} - \phi(\rho)]W(\rho)\, d\rho \quad (11)$$

Since the integral in (11) is non-negative and $\tau(r) \to 0$, we find at once that $\int_0^\infty \rho W(\rho)\, d\rho \geq 2l + 1$. To exclude equality we observe that there must exist two adjacent intervals $[\xi, \eta]$ and $[\eta, \zeta]$ $(\xi < \eta < \zeta < \infty)$ such that $\int_\xi^\eta \rho W(\rho)\, d\rho$ and $\int_\eta^\zeta \rho W(\rho)\, d\rho$ both exceed 1/4, say. If $r \geq \eta$, then, by (7), $r^{l+1} - \phi(r) \geq \int_\xi^\eta G(r, \rho)\phi(\rho)W(\rho)\, d\rho$ and throughout the interval $[\eta, \zeta]$ $(\rho^{l+1} - \phi(\rho))\rho^{-(l+1)} \geq c$, where c is some positive constant. For $r > \zeta$ we find from (11)

$$\int_0^r \rho W(\rho)\, d\rho \geq 2l + 1 - \tau(r) + c \int_\eta^\zeta \rho^l K(r, \rho)\rho W(\rho)\, d\rho$$

and in the limit $r \to \infty$

$$\int_0^\infty \rho W(\rho)\, d\rho \geq 2l + 1 + c \int^\zeta \rho W(\rho)\, d\rho > 2l + 1, \quad \text{q. e. d.}$$

(*d*) $\alpha > 0$, $\beta = \nu_n = \infty$. We start, as in (*b*), from equation 10, so that $G(r, \alpha) \geq \chi(r) > 0$ for $r > \alpha$. By assumption, $(2l + 1)^{-1}\theta(r) = \chi(r)/G(r, \alpha)$ approaches 0 as $r \to \infty$. From (10) we find

$$2l + 1 - \theta(r) - \int_\alpha^r \frac{H(r, \rho)}{G(r, \alpha)} \chi(\rho)W(\rho)\, d\rho = 0.$$

Hence, by (9),

$$\int_\alpha^r \rho W(\rho)\, d\rho \geq \int_\alpha^r \frac{H(r, \rho)}{G(r, \alpha)} G(\rho, \alpha)W(\rho)\, d\rho \geq 2l + 1 - \theta(r) + \int_\alpha^r \frac{H(r, \rho)}{G(r, \alpha)} [G(\rho, \alpha) - \chi(\rho)]W(\rho)\, d\rho \quad (12)$$

We proceed as in case (*c*) above. The inequality $\int_\alpha^\infty \rho W(\rho)\, d\rho \geq 2l + 1$ is an immediate consequence of (12). To exclude equality the intervals $[\xi, \eta]$ and $[\eta, \zeta]$ are chosen as before $(\xi \geq \alpha)$, so that $(2l + 1)(G(\rho, \alpha) - \chi(\rho))\rho^{-(l+1)} \geq c' > 0$ if $\eta \leq \rho \leq \zeta$. For $r > \zeta$ (12) implies

$$\int_\alpha^r \rho W(\rho)\, d\rho \geq 2l + 1 - \theta(r) + c' \int_\eta^\zeta \rho^l \frac{H(r, \rho)}{H(r, \alpha)} \rho W(\rho)\, d\rho$$

and since $\lim_{r\to\infty}(H(r, \rho)/H(r, \alpha)) = (\alpha/\rho)^l$, we find for $r \to \infty$ $\int_\alpha^\infty \rho W(\rho) d\rho \geq 2l + 1 + c'\alpha^l \int_\eta^\zeta \rho W(\rho)\, d\rho > 2l + 1$. This concludes the proof of (6).

5. *Examples.*—The proofs in the preceding section suggest the construction of potentials for which the inequalities (3) or (6) may be approxi-

mately replaced by the corresponding equalities. In 4(a), for example, the first inequality will nearly reduce to an equality if at those r where $W(r)$ is appreciable $\phi(r)$ nearly equals r^{l+1}, i.e., the field free solution. This leads (for $n_l = 1$) to the choice of a potential $V(r) = -W(r)$ $(W \geq 0)$ which vanishes everywhere with the exception of a small interval $a < r < a + \delta = b$. Outside $[a, b]$ we obtain for a suitably normalized solution of equation (5)

$$\left.\begin{aligned} \phi(r) &= (r/a)^{l+1} & r \leq a \\ \phi(r) &= c_1(b/r)^l - c_2(r/b)^{l+1} & r \geq b \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} (2l + 1)c_1 &= (l + 1)\phi(b) - b\phi'(b); \\ (2l + 1)c_2 &= -l\phi(b) - b\phi'(b) \end{aligned}\right\} \tag{14}$$

If $c_2 > 0$, $\phi(r) \to -\infty$ as $r \to \infty$, so that $\phi(r)$ vanishes at a point β given by $(\beta/b)^{2l+1} = c_1/c_2$, and if $c_2 = 0$, then $\beta = \infty$. Owing to the smallness of δ the relative change of $\phi(r)$ across the interval $[a, b]$ is negligible compared to the relative change of $\phi'(r)$, so that $\phi(b) \sim \phi(a)$. From the condition $c_2 \geq 0$ we obtain $-\phi'(b)/\phi(b) > l/b$, and since $\phi'(a)/\phi(a) = (l + 1)/a$, this amounts to

$$\phi'(a)/\phi(a) - \phi'(b)/\phi(b) \sim (\phi'(a) - \phi'(b))/\phi(a) \gtrsim (2l + 1)/a \tag{15}$$

Thus the required increment of the logarithmic derivative is the smaller the larger a is chosen—or a potential of given strength is the more effective in producing bound states the farther it is removed from the origin (which is the reason for the weight factor r in the integral I). Without yet specifying $W(r)$, we see from (5) that $\phi'(b) - \phi'(a)$ approximately equals $-\overline{W}\delta\phi(a)$ where $\overline{W}$ is a suitable average of W, provided the centrifugal term $l(l + 1)r^{-2}$ is negligible compared to $\overline{W}$. If the increment is as small as possible we find from (15) that $\overline{W}\delta a \sim (2l + 1)$ which is equivalent to $\int_a^b rW(r)\,dr = \int_0^\infty rW(r)\,dr \sim 2l + 1$.

To have a definite example consider $W(r) = 1 + l(l + 1)r^{-2}$, and $a = (2l + 1)(1 + \delta)/\delta$. Then, in $[a,b]$, $\phi(r) = \cos(r - a) + ((l + 1)/a) \sin(r - \alpha)$ so that $\phi(b) = \cos\delta + ((l + 1)/a)\sin\delta$, $\phi'(b) = -\sin\delta + ((l + 1)/a)\cos\delta$, and one verifies easily that $c_2 > 0$ for small δ (e.g., $\delta < 1/4$). The zero, β, is determined by $(\beta/b)^{2l+1} = c_1/c_2$, and approximately $c_1/c_2 \sim \delta^{-1}$ so that $\beta \sim a\cdot\delta^{-1/(2l+1)}$. Finally, $I = \int_a^b rW(r)\,dr = \delta(a + {}^1/{}_2\delta) + l(l + 1)\log(1 + \delta/a)$. As $\delta \to 0$, $I \to 2l + 1$. Alternately, instead of varying a and keeping the strength of W fixed, one might keep a fixed and vary the strength of the potential.

In a similar way one may construct potentials with two or more bound state solutions such that I is arbitrarily close to $(2l + 1)n_l$. One simply has to add other troughs in suitably placed intervals $[a', a' + \delta']$, etc., in such a way, however, that two successive intervals are sufficiently far from one another and from the zero of $\phi(r)$ between them. Note that these potentials are adjusted only to one fixed value of the angular momentum.

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[5] For $l = 0$ these statements are proved in the appendix of ref. 4. For higher l a similar proof may be given.

[6] Even for $l = 0$ this case is significant although no bound state is present. In particular in this case $|\sin \eta(0)| = 1$ where $\eta(k)$ is the scattering phase shift, and hence the cross section $\sin^2 \eta(k)/k^2$ becomes infinite as $k \to 0$. (See ref. 4, equation (1.9). In this case $f(0) = 0$.)

[7] For $l = 0$ one need not distinguish $\alpha = 0$ and $\alpha > 0$, because $G(r, \alpha) = r - \alpha$ may be used in both cases.

ON THE INVARIANT THEORY OF THE CLASSICAL GROUPS

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It has been recognized for some time that the theory of invariants and covariants, with respect to a given group, rests on the analysis of $\Gamma \otimes \Gamma'$, where Γ and Γ' are irreducible representations of the group, into its irreducible components. Thus if we denote by $\{\lambda\}$ the irreducible representation of the n-dimensional linear group which is associated with the partition $(\lambda) = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$, of any non-negative integer m into not more than n parts the core of the theory of invariants and covariants, under linear transformations, is the analysis of $\{\lambda\} \otimes \{\mu\}$ where (λ) and (μ) are partitions of any two non-negative integers m and j, respectively. The cases where (λ) is either the 1-element partition (m) or the m-element partition (1^m) and (μ) is either the 1-element partition (j) or the j-element partition (1^j) are of particular importance and the problem of analyzing $\{\lambda\} \otimes \{\mu\}$, especially in these cases, has been much studied, following the initial impetus given by Littlewood,[1] during the past decade. However the methods used have been laborious when m and j are greater than 2; in these cases $\{\lambda\} \otimes \{\mu\}$ contains many components, each corresponding to a partition of mj, and each of these has had to be determined separately by a tedious calculation. We present in this note a method which yields, in the cases of particular importance referred to, the components of $\{\lambda\} \otimes \{\mu\}$ in platoons, rather than individually, each platoon consisting of those parentheses $\{\ldots\}$ which contain the same number of non-zero parts.